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On some equilibrium problems for multimaps

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Abstract

In this paper, we first establish the continuity property for multimaps and generalized Berge's theorem for multimaps. Then we apply these results, and the Fan–Browder fixed point theorem to establish the existence theorems of quasi-equilibrium problems and generalized quasi-equilibrium problems for multimaps. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

By an equilibrium problem (shortly EP), as understood by Blum and Oettli [3] is the problem of finding

$$\text{(EP)} \quad \hat{x} \in X \text{ such that } f(\hat{x}, y) \geqslant 0 \text{ for all } y \in X,$$

where X is a given set and $f : X \times X \to \bar{R}$ is a given function with $f(x,x) = 0$.

A quasi-equilibrium problem (shortly QEP) is to find

$$\hat{x} \in S(\hat{x}) \text{ such that } f(\hat{x}, y) \geqslant 0 \text{ for all } y \in S(\hat{x}),$$

where X and f are as above and $S : X \multimap X$ is a given multimap with nonempty values.

A generalized quasi-equilibrium problem (shortly GQEP) is to find

$$\hat{x} \in S(\hat{x}) \quad \text{and} \quad \hat{y} \in T(\hat{x}) \quad \text{such that } \varphi(\hat{x}, \hat{y}, z) \geqslant 0$$

for all $z \in S(\hat{x})$, where X and S are the same as above, Y another given set, $T : X \multimap Y$ another multimap, and $\varphi : X \times Y \times X \to \bar{R}$ a given function.

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These types of problems contains variational inequalities, optimization problems, saddle points, Nash equilibria in noncooperative games, fixed points, convex differentiable optimization, complementary problems, for detail, see [2].

Our main aim in this paper is to establish the continuity properties for multimaps and a generalization of Berge's Theorem for multimaps. These results together with the Fan–Browder fixed point theorem [3] or a fixed point theorem due to Lin and Yu [11], give the existence theorems for the problem of finding

$$(\mathrm{QEP})' \quad \bar{x} \in S(\bar{x}) \text{ such that } z \notin -\mathrm{int}\, D \text{ for all } z \in F(\bar{x}, u) \text{ and all } u \in S(\bar{x}),$$

where $F : X \times X \multimap Z$ and $F(x,x) \subset D$ for all $x \in X$, Z is a real topological vector space (shortly t.v.s.), and D is a pointed closed convex cone in Z with $\mathrm{int}\, D \neq \emptyset$.

Use the existence theorems of $(\mathrm{QEP})'$ and a convex space version of continuous selection theorem due to Horvath [7, Theorem 3.2], we establish the existence theorems of the problem of finding

$$(\mathrm{GQEP})' \quad \bar{x} \in S(\bar{x}), \bar{y} \in T(\bar{x}) \text{ such that } z \notin -\mathrm{int}\, D \text{ for all } z \in F(\bar{x}, \bar{y}, u) \text{ and all } u \in S(\bar{x}),$$

where $F : X \times Y \times X \multimap Z$ and $F(x,y,x) \subset D$ for all $x \in X$, $y \in Y$, Z is a real t.v.s., and D is a pointed closed convex cone in Z with $\mathrm{int}\, D \neq \emptyset$. Consequently, we can drastically generalize and improve the results in [10,12] and many others.

In the concluding remarks, we will show that $(\mathrm{QEP})'$ and $(\mathrm{GQEP})'$ have some applications to the study of vector quasi-variational inequalities and generalized vector quasi-variational inequalities for fuzzy mappings.

2. Preliminaries

A multimap $T : X \multimap Y$ is a function from X into the power set of Y with nonempty values $T(x)$ for $x \in X$ and fibers $T^{-}(y)$ for $y \in Y$. Note that $x \in T^{-}(y)$ if and only if $y \in T(x)$.

For topological spaces X and Y, a multimap $T : X \multimap Y$ is said to be closed if its graph $G_r(T) = \{(x, y) \mid x \in X,\ y \in T(x)\}$ is closed in $X \times Y$, to have open fiber if $T^{-}(y)$ is open for all $y \in Y$; to be compact if the closure $\overline{T(X)}$ of its range $T(X)$ is compact in Y; to be upper semicontinuous (shortly u.s.c.) if for every $x \in X$ and every open set V in Y with $T(x) \subset V$, there exists a neighborhood $W(x)$ of x such that $T(W(x)) \subset V$; to be lower semicontinuous (shortly l.s.c.) if for every $x \in X$ and every open neighborhood $V(y)$ of every $y \in T(x)$, there exists a neighborhood $W(x)$ of x such that $T(u) \cap V(y) \neq \emptyset$ for all $u \in W(x)$; and to be continuous if it is both u.s.c. and l.s.c.

Let $A \subset X$, we denote $\mathrm{int}\, A$ and $\bar{A}$ to be the interior of A and the closure of A, respectively.

A real-valued function $g : X \to R$ on a topological space X is l.s.c. [resp. u.s.c.] if $\{x \in X \mid g(x) > r\}$ [resp. $\{x \in X \mid g(x) < r\}$] is open for each $r \in R$. A nonempty subset X of a t.v.s. E is said to be admissible (in the sense of Klee [8]) provided that, for every compact subset A of X and every neighborhood V of the origin O of E, there exists a continuous mapping $h : A \to X$ such that $x - h(x) \in V$ for all $x \in A$ and $h(A)$ is contained in a finite dimensional subspace L of E. Note that every nonempty convex subsets of a locally convex t.v.s. is admissible, l^p, $L^p(0,1)$ for $0 < p < 1$, the Hardy space H^p for $0 < p < 1$, locally convex subset of an F-normable t.v.s. and every compact convex locally convex subset of a t.v.s. are admissible. For detail see [14,6,17].

Recall that a nonempty topological space is acyclic if all of its reduced Čech homology groups over rational vanish. A multimap $T : X \multimap Y$ is said to be acyclic if it is u.s.c. with acyclic compact values. We denote

$$V(X, Y) = \{T \mid T : X \multimap Y \text{ is an acyclic multimap}\}.$$

Let X be a convex subset of a t.v.s. and Y a topological space. If $S, T : X \multimap Y$ are two multimaps such that $T(\operatorname{co} B) \subseteq S(B)$ for each finite subset B of X, then we call S a generalized KKM mapping w.r.t. T, where $\operatorname{co} B$ denotes the convex hull of B. A multimap $T : X \multimap Y$ is said to have the KKM property provided that for any generalized KKM mapping S w.r.t. T, the family $\{\overline{S(x)} : x \in X\}$ has the finite intersection property. We denote

$$\mathrm{KKM}(X, Y) = \{T \mid T : X \multimap Y \text{ has the KKM property}\}.$$

In [5], Chang showed that $V(X, Y) \subset \mathrm{KKM}(X, Y)$.

Definition 1. Let Z be a real t.v.s., D a convex cone in Z with $\operatorname{int} D \neq \emptyset$, and A a nonempty subset of Z.

(i) Let $y_1, y_2 \in A$, we denote

$$y_1 \leqslant y_2 \quad \text{if } y_2 - y_1 \in D,$$
$$y_1 < y_2 \quad \text{if } y_2 - y_1 \in \operatorname{int} D.$$

A point $\bar{y} \in A$ is called a vector minimal point of A if for any $y \in A$, $y - \bar{y} \notin -D \backslash \{0\}$.

A point $\bar{y} \in A$ is called a weakly vector minimal point of A if for any $y \in A$, $y - \bar{y} \notin -\operatorname{int} D$.

The set of vector minimal (resp. weakly vector minimal) points of A is denoted by $\mathrm{Min}_D A$ (resp. $w\,\mathrm{Min}_D A$).

(ii) A function $p : Z \to R$ is called a strictly monotonically increasing function if $a, b \in Z$ and $a - b \in \operatorname{int} D$, then $p(a) > p(b)$.

Definition 2. Let X be a nonempty convex subset of a real t.v.s. E, Z a real t.v.s., and D a convex cone in Z. Let $G : X \multimap Z$ be a multimap, G is said to be D-quasiconvex if for any $z \in Z$, the set

$$\{x \in X \mid \text{there is a } y \in G(x) \text{ such that } z - y \in D\} \text{ is convex.}$$

Throughout this paper, all topological space are assumed to be Hausdorff. In this paper, we need the following lemmas.

Lemma 1 (Luc [13]). *Let Z be a t.v.s., D a closed convex cone in Z such that $\operatorname{int} D \neq \emptyset$. Then, for any fixed $e \in \operatorname{int} D$ and $z \in Z$,*

$$p(y) = \min\{t \in R \mid y \in z + te - D\}$$

is a continuous and strictly monotonically increasing function from Z to R.

Lemma 2 (Aubin and Cellina [1]). *Let X and Y be two topological spaces and $T : X \multimap Y$ be a multimap*:

(i) *If X is compact and T is u.s.c. with compact values, then $T(X)$ is compact.*

(ii) *If Y is compact and T is closed, then T is u.s.c.*
(iii) *If T is an u.s.c. multimap with closed values, then T is closed.*

Lemma 3 (Luc [13]). *Let A be a nonempty compact subsets of a real t.v.s., D a closed convex cone in Z such that $D \neq Z$, then* $\mathrm{Min}_D A \neq \emptyset$.

Lemma 4 (Lin et al. [11]). *Let E be a t.v.s., X an admissible convex subset of E, and $T \in \mathrm{KKM}(X,X)$ be compact and closed. Then T has a fixed point $\bar{x} \in X$.*

From the definition of l.s.c., it is easy to obtain the following lemmas.

Lemma 5 (Klein et al. [9]). *Let X and Y be topological spaces, $T : X \multimap Y$ be l.s.c. Then $\bar{T} : X \multimap Y$ is l.s.c., where $\bar{T} : X \multimap Y$ is defined by $\bar{T}(x) = \overline{T(x)}$ for $x \in X$.*

Lemma 6 (Klee [8]). *Let X be a convex subset of a t.v.s. E, Z a t.v.s., D a closed convex cone in Z such that* $\mathrm{int}\, D \neq \emptyset$. *Let $F : X \multimap Z$ be a multimap. For any fixed $e \in \mathrm{int}\, D$ and any fixed $a \in Z$, let*

$$p(y) = \min\{t \in R : y \in a + te - D\}.$$

If F is D-quasiconvex, then $pF : X \multimap R$ is R^+-quasiconvex.

Lemma 7 (Horvath [7]). *Let X be a paracompact space, Y a convex space, $G, T : X \multimap Y$ two multimaps such that for any $x \in X$, $\mathrm{co}\, G(x) \subset T(x)$ and $X = \bigcup\{\mathrm{int}\, G^-(y) : y \in Y\}$. Then T has a continuous selection $f : X \to Y$; that is, $f(x) \in T(x)$ for each $x \in X$.*

Definition 3. Let X be a topological space, Y be a t.v.s. A function $f : X \to Y$ is said to be demicontinuous if

$$f^-(M) = \{x \in X \mid f(x) \in M\}$$

is closed in X for each closed half space $M \subset Y$.

Lemma 8 (Tanaka [16]). *Let X be a topological space, Z a t.v.s. and $f : X \to Z$ be a demicontinuous function, then for any $\varphi \in Z^*$, the composite mapping $\varphi \circ f$ is continuous, where $Z^* = \{\varphi \mid \varphi : z \to R$ is a linear continuous function$\}$.*

Lemma 9 (Aubin and Cellina [1]). *Let X, Y be topological spaces, $F, G : X \multimap Y$ such that $F(x) \cap G(x) \neq \emptyset$ for all $x \in X$. Suppose that $x_0 \in X$ and*
(i) *F is u.s.c. at x_0;*
(ii) *$F(x_0)$ is compact; and*

(iii) *G is closed.*
Then the multimap $F \cap G : x \multimap F(x) \cap G(x)$ *is u.s.c. at* x_0.

Lemma 10 (Tan et al. [15]). *Let* X *be a topological space, and* $F : X \to R$ *be a continuous multimap with compact values. Then the function* $m : X \to R$ *defined by* $m(x) = \max F(x)$ *is continuous.*

3. Continuity and Berge's theorem for multimaps

In [12], Lin established the following lemma.

Lemma A (Lin [12]). *Let* E_1, E_2 *and* Z *be real t.v.s.,* X *and* Y *be nonempty subsets of* E_1 *and* E_2, *respectively, and* $F: X \times Y \multimap Z$ *and* $S: Y \multimap X$ *be continuous multimaps with compact values. Then* $T: Y \multimap Z$ *defined by* $T(y) = \bigcup_{x \in S(y)} F(x, y) = F(S(y), y)$ *is continuous.*

In this paper we need a reformulation of Lemma A stated here as Theorem 1, and its proof is also given for completeness.

Theorem 1. *Let* E_1, E_2 *and* Z *be real t.v.s.,* X *and* Y *be nonempty subsets of* E_1 *and* E_2 *respectively. Let* $F: X \times Y \multimap Z$, $S: X \multimap Y$ *be multimaps*:
(a) *If both* S *and* F *are l.s.c., then* $T : X \multimap Z$ *defined by*

$$T(x) = \bigcup_{y \in S(x)} F(x, y) = F(x, S(x)) \text{ is l.s.c. on } X.$$

(b) *If both* S *and* F *are u.s.c. with compact values, then* T *is an u.s.c. multimap with compact values.*

Proof. (a) Under the assumption of (a), we show that T is l.s.c. at x. Let $x \in X$ and G be any open set in Z with $T(x) \cap G \neq \emptyset$. Then there is a $u \in S(x)$ such that $F(x, u) \cap G \neq \emptyset$. Since F is l.s.c. at (x, u), there exist open neighborhoods $N(x)$ of x in X and $N_x(u)$ of u in Y such that for any $x' \in N(x)$, $u' \in N_x(u)$, $F(x', u') \cap G \neq \emptyset$. Since $u \in N_x(u) \cap S(x) \neq \emptyset$ and S is l.s.c. at x, there exists an open neighborhood $O_0(x)$ of x in X such that $N_x(u) \cap S(x') \neq \emptyset$ for any $x' \in O_0(x)$. Let $N_1(x) = O_0(x) \cap N(x)$. If $x' \in N_1(x)$, $u' \in N_x(u) \cap S(x') \neq \emptyset$, then $F(x', u') \cap G \neq \emptyset$. Therefore, $T(x') \cap G = \bigcup_{u \in S(x')} F(x', u) \cap G \neq \emptyset$ for all $x' \in N_1(x)$. This shows that T is l.s.c. at x and T is l.s.c. on X.

(b) Under the assumption (b), we show that T is u.s.c. on X. Let $x \in X$ and G be any open set in Z with $T(x) \subset G$. Then $F(x, u) \subset G$ for all $u \in S(x)$. Since F is u.s.c. at (x, u), there exist open neighborhoods $O(u)$ of u in Y and $O_u(x)$ of x in X such that $F(x', u') \subset G$ for any $u' \in O(u)$ and $x' \in O_u(x)$. Since $S(x) \subset \bigcup_{u \in S(x)} O(u)$ and $S(x)$ is compact, there exists $\{u_1, u_2, \ldots, u_n\} \subset S(x)$ such that $S(x) \subset \bigcup_{i=1}^n O(u_i)$. By assumption (b), S is u.s.c. at x, there is an open neighborhood $O_0(x)$ of x such that for all $x' \in O_0(x)$, $S(x') \subset \bigcup_{i=1}^n O(u_i)$. Let $O_1(x) = O_0(x) \cap (\bigcap_{i=1}^n O_{u_i}(x))$, then $O_1(x)$ is an open neighborhood of x. If $x' \in O_1(x)$, and $u' \in S(x')$, then there exists $j \in \{1, 2, \ldots, n\}$ such that $u' \in O(u_j)$. Since $x' \in O_{u_j}(x)$, $F(x', u') \subset G$ and $T(x') = \bigcup_{u' \in S(x')} F(x', u') \subset G$ for all $x' \in O_1(x)$. This shows that T is u.s.c. at x. Since F is u.s.c. with compact values and $S(x)$ is compact for all $x \in X$, it follows from Lemma 2 that $T(x) = F(x, S(x))$ is a compact set. $\square$

As a consequence of Theorem 1, we establish the following theorem.

Theorem 2. *Let E_1, E_2 and Z be t.v.s., X and Y be nonempty subsets of E_1 and E_2, respectively, $F : X \times Y \times X \multimap Z$ and $S : X \multimap X$:*
(a) *If both S and F are l.s.c., then $T : X \times Y \multimap Z$ defined by*

$$T(x,y) = \bigcup_{u \in S(x)} F(x,y,u) = F(x,y,S(x))$$

is l.s.c. on $X \times Y$.
(b) *If both S and F are u.s.c. with compact values, then T is an u.s.c. multimap with compact values.*

Proof. Let the multimap $H : X \times Y \multimap X$ be defined by

$$H(x,y) = S(x) \quad \text{for } (x,y) \in X \times Y.$$

It is easy to see that if $S : X \multimap X$ is l.s.c., then $H : X \times Y \multimap X$ is l.s.c. and if S is u.s.c., then H is u.s.c. We also see that

$$T(x,y) = \bigcup_{u \in S(x)} F(x,y,u) = \bigcup_{u \in H(x,y)} F(x,y,u).$$

Then the conclusion of Theorem 2 follows from Theorem 1. □

Applying Theorem 1, we obtain a generalized Berge's theorem for multimaps.

Theorem 3. *Let X and Y be topological spaces, $F : X \times Y \multimap R$, $S : X \multimap Y$, $m(x) = \sup F(x,S(x))$, and $M(x) = \{y \in S(x) : m(x) \in F(x,y)\}$:*
(a) *If both F and S are l.s.c., then m is l.s.c.*
(b) *If both F and S are u.s.c. with compact values, then m is u.s.c.*
(c) *If both F and S are continuous multimaps with compact values, then m is a continuous function and M is an u.s.c. closed multimap.*

Proof. (a) Let $T(x) = F(x,S(x))$, then by assumption (a) and Theorem 1, T is l.s.c. Let $t \in R$, and $M_t = \{u \in X : m(u) > t\}$, then M_t is an open set. Indeed, let $x \in M_t$, then either $t < m(x) = \infty$ or $t < m(x) < \infty$. In any case, there exists a $z \in T(x)$ such that $z > t$. Let $0 < \varepsilon < z - t$. Since $(z - \varepsilon,\ z + \varepsilon) \cap T(x) \neq \emptyset$ and T is l.s.c., there exists an open neighborhood $V(x)$ of x such that

$$(z - \varepsilon,\ z + \varepsilon) \cap T(x') \neq \emptyset \quad \text{for all } x' \in V(x).$$

Let $z' \in (z - \varepsilon, z + \varepsilon) \cap T(x')$ for $x' \in V(x)$. Then $z' > z - \varepsilon > t$ and $z' \in T(x')$ for $x' \in V(x)$. Therefore, $m(x') = \sup T(x') \geqslant z' > t$ for all $x' \in V(x)$. This shows that $x' \in M_t$ for all $x' \in V(x)$ and $V(x) \subset M_t$. Hence M_t is open. Therefore m is l.s.c.

(b) By assumption and Theorem 1, $T(x) = F(x,S(x))$ is u.s.c. with compact values. Therefore, $T(x)$ is bounded and $m(x) < \infty$ for all $x \in X$. Let ε be any positive number, since $T(x) \subset (-\infty, m(x) + \varepsilon)$ and T is u.s.c. at x, there exists an open neighborhood $O_2(x)$ of x in X such that

$$T(x') \subset (-\infty, m(x) + \varepsilon) \quad \text{for any } x' \in O_2(x).$$

Therefore $m(x') = \sup T(x') \leqslant m(x) + \varepsilon$ for any $x' \in O_2(x)$. This shows that m is u.s.c. at x.

(c) The continuity of m follows from (a) and (b). M is closed. Indeed, let $\{(x_\alpha, y_\alpha)\}_{\alpha\in\Lambda}$ be a net in G_rM such that $(x_\alpha, y_\alpha) \to (\bar{x}, \bar{y}) \in X \times Y$. Then $y_\alpha \in S(x_\alpha)$ and $m(x_\alpha) \in F(x_\alpha, y_\alpha)$. By Lemma 2, F and S are closed. Since m is continuous, $m(x_\alpha) \to m(\bar{x})$, $\bar{y} \in S(\bar{x})$ and $m(\bar{x}) \in F(\bar{x}, \bar{y})$. Therefore $\bar{y} \in M(\bar{x})$ and $(\bar{x}, \bar{y}) \in G_rM$. This shows that M is closed. Let $H(x) = \{y \in Y:\ m(x) \in F(x, y)\}$, then $M(x) = S(x) \cap H(x)$. With the same argument as for M, we can show that H is closed. By assumption (c), S is u.s.c. with compact values, it follows from Lemma 9 that M is an u.s.c. closed multimap.

If F is a single-valued function, then we have the following corollary.

Corollary 1. *Let X and Y be topological spaces, $f : X\times Y \to R$, $S : X \multimap Y$, $m(x)=\sup_{y\in S(x)} f(x, y)$, and*

$$M(x) = \{y \in S(x) : m(x) = f(x, y)\}.$$

If S is a continuous multimap with compact values and f is a continuous function, then m is a continuous function and M is an u.s.c. closed multimap.

Proof. For each $x \in X$, let $F(x, y) = \{f(x, y)\}$, then $F : X \times Y \multimap R$ is a continuous multimap with compact values. The conclusion follows from Theorem 3.

The following theorem is a basic tool in the proof of equilibrium theorems of this paper.

Theorem 4. *Let X be a nonempty subset of topological space E_1, Z a real t.v.s. and D a closed pointed convex cone with* $\operatorname{int} D \neq \emptyset$ *and $D \neq Z$, $S : X \multimap X$ and $F : X \times X \multimap Z$. Let $m : X \multimap Z$ be defined by*

$$m(x) = w \operatorname*{Min}_D F(x, S(x)) \quad for\ x \in X$$

and $M : X \multimap X$ be defined by

$$M(x) = \{u \in S(x):\ F(x, u) \cap m(x) \neq \emptyset\}.$$

If both F and S are compact continuous multimaps with closed values, then both M and m are closed compact u.s.c. multimaps.

Proof. (1) Since F is u.s.c. with compact values, and $S(x)$ is compact for all $x \in X$, it follows from Lemma 2 that $F(x, S(x))$ is a compact set. By Lemma 3, $\phi \neq \operatorname{Min}_D F(x, S(x)) \subset w \operatorname{Min}_D F(x, S(x))$. Hence $m(x) \neq \emptyset$ for all $x \in X$. m is closed. Indeed, let $\{(x_\alpha, z_\alpha)\}$ be a net in $X \times Z$ such that $(x_\alpha, z_\alpha) \to (\bar{x}, \bar{z}) \in X \times Z$ and $z_\alpha \in m(x_\alpha)$ for all α, then there exists a $u_\alpha \in S(x_\alpha)$ such that $z_\alpha \in F(x_\alpha, u_\alpha)$ and

$$z_\alpha \in w \operatorname*{Min}_D F(x_\alpha, S(x_\alpha)) = m(x_\alpha) \subseteq \overline{F(X \times X)}.$$

By assumption, $\overline{S(X)}$ is compact. We may assume that $u_\alpha \to \bar{u} \in \overline{S(X)}$. Since F and S are u.s.c. with closed values, it follows from Lemma 2 that F and S are closed. Therefore $\bar{u} \in S(\bar{x})$ and $\bar{z} \in F(\bar{x}, \bar{u})$. We want to show that $\bar{z} \in m(\bar{x})$. Suppose that $\bar{z} \notin m(\bar{x}) = w \operatorname{Min}_D F(\bar{x}, S(\bar{x}))$, then there exist $s^* \in S(\bar{x})$, $w \in F(\bar{x}, s^*)$ such that $w - \bar{z} \in -\operatorname{int} D$. By Theorem 1, $x \mapsto F(x, S(x))$ is l.s.c.

Since $x_\alpha \to \bar{x}$, there exists a net $w_\alpha \in F(x_\alpha, S(x_\alpha))$ such that $w_\alpha \to w$. Therefore, $w_\alpha - z_\alpha \in -\operatorname{int} D$ for sufficiently large α. This is impossible since $z_\alpha \in m(x_\alpha) = w\operatorname{Min}_D F(x_\alpha, S(x_\alpha))$. This shows that $\bar{z} \in m(\bar{x})$ and m is closed. Since $m(x) \subseteq F(x, S(x))$ for all $x \in X$ and F is compact, it follows that m is compact. By Lemma 2, m is u.s.c. Therefore m is a compact u.s.c. closed multimap.

(2) M is closed. Indeed, let $\{(x_\alpha, u_\alpha)\}_{\alpha \in J}$ be a net in $X \times X$ such that $(x_\alpha, u_\alpha) \to (\bar{x}, \bar{u}) \in X \times X$ and $u_\alpha \in M(x_\alpha)$ for all $\alpha \in J$. Then $u_\alpha \in S(x_\alpha)$ and $F(x_\alpha, u_\alpha) \cap m(x_\alpha) \neq \emptyset$. Let $z_\alpha \in F(x_\alpha, u_\alpha) \cap m(x_\alpha)$. By (1), we see that S, m and F are compact and closed, and we may assume that $z_\alpha \to \bar{z} \in Z$. Hence $\bar{u} \in S(\bar{x})$ and $\bar{z} \in F(\bar{x}, \bar{u}) \cap m(\bar{x})$. This shows that $\bar{u} \in M(\bar{x})$ and M is closed. Since S is compact, M is compact. By Lemma 2, M is u.s.c.. This completes the proof of the theorem. □

Corollary 2 (Lin [12]). *Let X and Y be nonempty subsets of topological spaces E_1 and E_2, respectively, and Z a real t.v.s., D a closed pointed convex cone in Z with $\operatorname{int} D \neq \emptyset$ and $D \neq Z$, $S: X \multimap X$ and $F: X \times Y \times X \multimap Z$. Let $m: X \multimap Z$ be defined by $m(x, y) = w\operatorname{Min}_D F(x, y, S(x))$ for $(x, y) \in X \times Y$ and $M: X \times Y \multimap X$ be defined by*

$$M(x, y) = \left\{ u \in S(x) : F(x, y, u) \cap w \operatorname*{Min}_D F(x, y, S(x)) \neq \emptyset \right\}.$$

If both F and S are compact continuous multimaps with closed values, then both M and m are closed compact u.s.c. multimaps.

Proof. Apply Theorem 4 and use the same argument as for Theorem 2. □

4. Equilibrium theorems of multimaps

As consequences of Theorem 4 and Lemma 4, we obtain the following equilibrium theorems.

Theorem 5. *Let X be an admissible convex subset of a t.v.s. E, $S : X \multimap X$ a compact continuous multimap with nonempty closed values, and Z a real t.v.s., D a closed pointed convex cone in Z with $\operatorname{int} D \neq \emptyset$ and $D \neq Z$. Let $F : X \times X \multimap Z$ be a continuous multimap with compact values. Let $M : X \multimap X$ be defined by $M(x) = \{y \in S(x) : F(x, y) \cap w\operatorname{Min}_D F(x, S(x)) \neq \emptyset\}$ for $x \in X$. Suppose that $M \in \mathrm{KKM}(X, X)$. Then there exists an $\bar{x} \in X$, such that $\bar{x} \in S(\bar{x})$ and $F(\bar{x}, \bar{x}) \cap w\operatorname{Min}_D F(\bar{x}, S(\bar{x})) \neq \emptyset$. Furthermore, if $F(x, x) \subset D$ for all $x \in X$, then there exists an $\bar{x} \in S(\bar{x})$ such that*

$$z \notin -\operatorname{int} D \quad \textit{for all } z \in F(\bar{x}, u) \textit{ and all } u \in S(\bar{x}).$$

Proof. It follows from Theorem 4 that M is a compact closed multimap. By assumption, $M \in \mathrm{KKM}(X, X)$. It follows from Lemma 4 that there exists an $\bar{x} \in X$ such that $\bar{x} \in M(\bar{x})$. This shows that $\bar{x} \in S(\bar{x})$ and $F(\bar{x}, \bar{x}) \cap w\operatorname{Min}_D F(\bar{x}, S(\bar{x})) \neq \emptyset$. Therefore, there exists a $\bar{z} \in F(\bar{x}, \bar{x})$ such that $z - \bar{z} \notin -\operatorname{int} D$ for all $z \in F(\bar{x}, u)$ and all $u \in S(\bar{x})$.

In particular, if $F(x, x) \subset D$ for all $x \in X$, then $z \notin -\operatorname{int} D$ for all $z \in F(\bar{x}, u)$ and all $u \in S(\bar{x})$. Indeed, if there exist a $v \in S(\bar{x})$ and a $w \in F(\bar{x}, u)$ such that $w \in -\operatorname{int} D$, then $w - \bar{z} \in -\operatorname{int} D - D \subset -\operatorname{int} D$. This leads to a contradiction with $z - \bar{z} \notin -\operatorname{int} D$ for all $z \in F(\bar{x}, u)$ and all $u \in S(\bar{x})$. Therefore $z \notin -\operatorname{int} D$ for all $z \in F(\bar{x}, u)$ and all $u \in S(\bar{x})$.

Corollary 3. *Let X be an admissible convex subset of a t.v.s. E, $S : X \multimap X$ a compact continuous multimap with nonempty closed values, Z a real t.v.s., D a closed pointed convex cone in Z with* $\operatorname{int} D \neq \emptyset$ *and $D \neq Z$. Let $f : X \times X \to Z$ be a continuous function and $M : X \multimap X$ be defined by*

$$M(x) = \left\{ y \in S(x) : f(x,y) \in w \operatorname*{Min}_D f(x, S(x)) \right\} \quad \text{for } x \in X.$$

Suppose that $M \in \mathrm{KKM}(X,X)$, then there exists an $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$ and $f(\bar{x},\bar{x}) \in w \operatorname{Min}_D f(x, S(x))$. Furthermore, if $f(x,x) \in D$ for all $x \in X$, then there exists an $\bar{x} \in S(\bar{x})$ such that $f(\bar{x},u) \notin -\operatorname{int} D$ for all $u \in S(\bar{x})$.

Proof. The result follows immediately from Theorem 5. □

In order to show that the generalized quasi-equilibrium theorem can be reduced from the quasi-equilibrium theorem, we need the following theorem.

Theorem 6. *Let X be an admissible convex subset of a t.v.s. E_1, E_2 a t.v.s., $Y \subset E_2$, $S : X \multimap X$ a compact continuous multimap with nonempty closed values, $f : X \to Y$ a continuous function, Z a real t.v.s., D a closed pointed convex cone in Z with* $\operatorname{int} D \neq \emptyset$ *and $D \neq Z$. Let $F : X \times Y \times X \multimap Z$ a continuous multimap with compact values. Let $H : X \multimap X$ be defined by*

$$H(x) = \left\{ u \in S(x) \,|\, F(x, f(x), u) \cap w \operatorname*{Min}_D F(x, f(x), S(x)) \neq \emptyset \right\}$$

for $x \in X$. Suppose that $H \in \mathrm{KKM}(X,X)$, then there exist $(\bar{x},\bar{y}) \in X \times Y$, $\bar{x} \in S(\bar{x})$, $\bar{y} = f(\bar{x})$, $\bar{z} \in F(\bar{x},\bar{y},\bar{x})$ such that $z - \bar{z} \notin -\operatorname{int} D$ for all $z \in F(\bar{x},\bar{y},u)$ and all $u \in S(\bar{x})$.

In particular, if $F(x,y,x) \subset D$ for all $(x,y) \in X \times Y$, then $z \notin -\operatorname{int} D$ for all $z \in F(\bar{x},\bar{y},u)$ and all $u \in S(\bar{x})$.

Proof. Let $G : X \times X \multimap Z$ be defined by $G(x,u) = F(x, f(x), u)$ for $(x,u) \in X \times X$. Then $H(x) = \{u \in S(x) \,|\, G(x,u) \cap w \operatorname{Min}_D G(x, S(x)) \neq \emptyset\}$. By assumption, $G : X \times X \multimap Z$ is a continuous multimap. Then it follows from Theorem 4 that H is a closed compact multimap. By Theorem 5, there exists an $\bar{x} \in H(\bar{x})$, that is, $\bar{x} \in S(\bar{x})$, and $\bar{z} \in F(\bar{x}, f(\bar{x}), \bar{x})$ such that $z - \bar{z} \notin -\operatorname{int} D$, for all $z \in F(\bar{x}, f(\bar{x}), u)$ and all $u \in S(\bar{x})$. Let $\bar{y} = f(\bar{x})$. This proves the first part of the theorem. The second part of Theorem 6 follows from the same argument as for Theorem 5.

Applying Theorem 6, we establish the existence theorem of generalized vector quasi-equilibrium problems.

Corollary 4. *Let E_1, E_2, X, Y, f, F, Z and D be the same as Theorem 6. Let $M : X \times Y \multimap X$ be defined by*

$$M(x,y) = \left\{ u \in S(x) \,|\, F(x,y,u) \cap w \operatorname*{Min}_D F(x,y,S(x)) \neq \emptyset \right\}$$

for $(x,y) \in X \times Y$. Suppose that for each $(x,y) \in X \times Y$, $M(x,y)$ is an acyclic set. Then, the conclusion of Theorem 6 holds.

Proof. Let $G: X \times X \multimap Z$ be defined by $G(x,u) = F(x, f(x), u)$ for $(x,u) \in X \times X$ and $H : X \multimap X$ be defined by

$$H(x) = M(x, f(x)) \quad \text{for } x \in X.$$

Then

$$H(x) = \left\{ u \in S(x) \mid G(x,u) \cap w \operatorname*{Min}_D G(x, S(x)) \neq \emptyset \right\}.$$

By assumption, $G : X \times X \multimap Z$ is a continuous multimap. Then, it follows from Theorem 4 that H is a closed compact multimap. By Lemma 2 and assumption, we see that H is a u.s.c. multimap with compact acyclic values. Therefore, $H \in V(X,X) \subset \mathrm{KKM}(X,X)$, and the conclusion follows from Theorem 6. □

By using a continuous selection theorem of Horvath [7] and Theorem 4, we show that a generalized quasi-equilibrium theorem is a consequence of a quasi-equilibrium theorem.

Theorem 7. *Let X be a convex subset of a locally convex t.v.s. E_1, E_2 a t.v.s. and $Y \subset E_2$ a convex subset, $S : X \multimap X$ a compact continuous multimap with nonempty closed values, $T : X \multimap Y$ a multimap with convex values, and $X = \bigcup_{y \in Y} \operatorname{int} T^-(y)$, Z a real t.v.s., D a closed pointed convex cone in Z with $\operatorname{int} D \neq \emptyset$ and $D \neq Z$. Let $F : X \times Y \times X \multimap Z$ be a continuous multimap with compact values. Suppose that for each $(x,y) \in X \times Y$, $M(x,y)$ is an acyclic set, where $M : X \times Y \multimap X$ is defined by*

$$M(x,y) = \left\{ u \in S(x) \mid F(x,y,u) \cap w \operatorname*{Min}_D F(x,y,S(x)) \neq \emptyset \right\} \quad for\ (x,y) \in X \times Y.$$

Then there exist $(\bar{x}, \bar{y}) \in X \times Y$, $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$, $\bar{z} \in F(\bar{x},\bar{y},\bar{x})$ such that $z - \bar{z} \notin -\operatorname{int} D$ for all $z \in F(\bar{x},\bar{y},u)$ and all $u \in S(\bar{x})$. In particular, if $F(x,y,x) \subset D$ for all $(x,y) \in X \times Y$, then $z \notin -\operatorname{int} D$ for all $z \in F(\bar{x},\bar{y},u)$ and all $u \in S(\bar{x})$.

Proof. Since $S : X \multimap X$ is compact, there exists a compact set $K \subset X$ such that $S(X) \subset K$. Let $X' = \operatorname{co} K$, then X' is a paracompact set. By Lemma 7, $T|_{X'}$ has a continuous selection $f : X' \to Y$. Since X' is a nonempty convex subset of a locally convex t.v.s., X' is admissible.

Apply Corollary 4 with X' instead of X, we show that there exist $\bar{x} \in S(\bar{x})$, $\bar{y} = f(\bar{x}) \in T(\bar{x})$, $\bar{z} \in F(\bar{x},\bar{y},\bar{x})$ such that $z - \bar{z} \notin -\operatorname{int} D$ for all $z \in F(\bar{x},\bar{y},u)$ and all $u \in S(\bar{x})$. □

Theorem 8. *Let X be a compact convex subset of a Banach space E_1 and Z be a real Banach space, D a closed pointed convex cone in Z with $\operatorname{int} D \neq \emptyset$ and $D \neq Z$, $F : X \times X \multimap Z$, $S : X \multimap X$ be a multimap such that for all $x \in X$, $S(x)$ is a nonempty convex set and for all $y \in Y$, $S^-(y)$ is an open set. Let $\bar{S} : X \multimap X$ be an u.s.c. multimap satisfying*

(i) *$F(x,x) \subset D$ for all $x \in X$;*
(ii) *F is a continuous multimap with compact values;*
(iii) *for each $x \in X$, $y \mapsto F(x,y)$ is D-quasiconvex.*

Then, there exists an $\bar{x} \in S(\bar{x})$, such that

$$z \notin -\operatorname{int} D \quad for\ all\ z \in F(\bar{x},u)\ and\ all\ u \in S(\bar{x}).$$

Proof. It follows from Lemma 1 that there exists a continuous and strictly monotonically increasing function p from Z to R. Fixing an arbitrary $n \in N$, we define $H_n : X \multimap X$ by

$$H_n(x) = \left\{ u \in S(x) \,|\, \min pF(x,u) < \min pF(x,\bar{S}(x)) + \frac{1}{n} \right\}.$$

Since $\bar{S}$ is a u.s.c. multimap with closed values, it follows from Lemma 2 that $\bar{S}$ is closed. Since X is compact and pF is a u.s.c. multimap with compact values, for each $x \in X$, $\bar{S}(x)$ and $pF(x,\bar{S}(x))$ are compact sets. Therefore, for each $x \in X$, $H_n(x)$ is a nonempty set. Indeed, there exists a $u \in \bar{S}(x) = \overline{S(x)}$ and $w \in pF(x,u)$ such that $w = \min pF(x,\bar{S}(x)) = \min pF(x,u)$. Since $u \in \overline{S(x)}$, there exists a net $u_\alpha \in S(x)$ such that $u_\alpha \mapsto u$. By Lemma 10, the function $(x,u) \mapsto \min pF(x,u)$ is continuous. Therefore, for each $n \in N$, $\min pF(x,u_\alpha) - \min pF(x,u) < 1/n$ for sufficiently large α. This shows that $\min pF(x,u_\alpha) < \min pF(x,u) + 1/n = \min pF(x,\bar{S}(x)) + 1/n$ and $u_\alpha \in H_n(x)$ for sufficiently large α. Since $u \multimap F(x,u)$ is D-quasiconvex, it follows from Lemma 6 that $u \multimap pF(x,u)$ is R^+-quasiconvex. Since $u \multimap pF(x,u)$ is R^+-quasiconvex and $S(x)$ is convex, it is easy to see that $H_n(x)$ is convex for each $x \in X$. Furthermore,

$$H_n^-(u) = S^-(u) \cap \left\{ x \in X \,|\, \min pF(x,u) < \min pF(x,\bar{S}(x)) + \frac{1}{n} \right\}.$$

By assumption, $S : X \multimap X$ is a multimap with open fibers, so S is l.s.c. By Lemma 5, $\bar{S}$ is also l.s.c. By assumption, $\bar{S} : X \multimap X$ is u.s.c. This shows that $\bar{S}$ is continuous. By Theorem 3 and Lemma 10, we see that for each $x \in X$, the mappings $x \mapsto \min pF(x,\bar{S}(x))$ and $(x,u) \mapsto \min pF(x,u)$ are continuous. Since S has open fibers, $H_n^-(u)$ is open in X. Now by Fan-Browder fixed point theorem [3], there exists an $\bar{x}_n \in X$ such that $\bar{x}_n \in H_n(\bar{x}_n)$, i.e., $\bar{x}_n \in S(\bar{x}_n)$,

$$\min pF(\bar{x}_n,\bar{x}_n) < \min pF(\bar{x}_n, S(\bar{x}_n)) + \frac{1}{n}.$$

Let $\bar{z}_n \in F(\bar{x}_n,\bar{x}_n)$ satisfying $p(\bar{z}_n) = \min pF(\bar{x}_n,\bar{x}_n)$. Since $\bar{S}$ is compact, and F is u.s.c. with compact values, it follows from Lemma 2 that $F(X,\bar{S}(X))$ is compact. Since $\bar{x}_n \in S(\bar{x}_n) \subset \bar{S}(X)$, $\bar{z}_n \in F(X,\bar{S}(X))$, there exist subsequence $\{x_{n(\alpha)}\}_{\alpha\in\Gamma}$ of $\{x_n\}$ and $\{z_{n(\alpha)}\}_{\alpha\in\Gamma}$ of $\{z_n\}$, $\bar{x} \in \bar{S}(X)$ and $\bar{z} \in F(X,\bar{S}(X))$ such that $\bar{x}_{n(\alpha)} \to \bar{x}$ and $\bar{z}_{n(\alpha)} \to \bar{z}$. For each $\alpha \in \Gamma$,

$$p(\bar{z}_{n(\alpha)}) < \min pF(\bar{x}_{n(\alpha)}, \bar{S}(\bar{x}_{n(\alpha)})) + \frac{1}{n(\alpha)}.$$

Letting $n(\alpha) \to \infty$, then by the continuity of the function $x \to \min pF(x,\bar{S}(x))$, we have

$$p(\bar{z}) \leqslant \min pF(\bar{x},\bar{S}(\bar{x})). \tag{1}$$

Since $\bar{S}$ and F are u.s.c. with compact values, it follows from Lemma 2 that F and $\bar{S}$ are closed. Since $\bar{x}_n \in \bar{S}(\bar{x}_n)$ and $\bar{z}_n \in F(\bar{x}_n,\bar{x}_n)$, we see that $\bar{x} \in \bar{S}(\bar{x})$, $\bar{z} \in F(\bar{x},\bar{x}) \subseteq F(\bar{x},\bar{S}(\bar{x}))$. Relation (1) shows that $p(\bar{z}) = \min pF(\bar{x},\bar{S}(\bar{x}))$. Therefore, $p(\bar{z}) \leqslant p(z)$ for all $z \in F(\bar{x},u)$ and for all $u \in \bar{S}(\bar{x})$. Since p is strictly monotonic increasing,

$$z - \bar{z} \notin -\operatorname{int} D \quad \text{for all } z \in F(\bar{x},u) \text{ and all } u \in \bar{S}(\bar{x}).$$

Since $F(x,x) \subset D$ for all $x \in X$, it is easy to see that $z \notin -\operatorname{int} D$ for all $z \in F(\bar{x},u)$ and all $u \in \bar{S}(\bar{x})$.

If F in Theorem 8 is a single-valued function, we obtain the following corollary.

Corollary 5. *Let X be a compact convex subset of Banach space E_1, Z be a real Banach space, D a closed pointed convex cone in Z with $\operatorname{int} D \neq \emptyset$ and $D \neq Z$. Let $S : X \multimap X$ be a multimap such that for all $x \in X$, $S(x)$ is a nonempty convex set, and for all $y \in X$, $S^{-}(y)$ is an open set. Let $\bar{S} : X \multimap X$ be an u.s.c. multimap, and let $f : X \times X \to Z$ be a function satisfying*

(i) *$f(x,x) \in D$ for all $x \in X$;*
(ii) *f is a continuous function;*
(iii) *for each $x \in X$, $y \to f(x,y)$ is D-quasiconvex.*

Then there exists an $\bar{x} \in X$, $\bar{x} \in S(\bar{x})$ such that $f(\bar{x},y) \notin -\operatorname{int} D$ for all $y \in S(\bar{x})$.

As a consequence of Theorem 7, we obtain the following theorem.

Theorem 9. *Let X, Y, Z, D, S and T be the same as Theorem 7, $f : X \times Y \times X \to Z$ be a demicontinuous function, $\varphi \in Z^{*}\setminus\{0\}$. Suppose that for each $(x,y) \in X \times Y$, the set $M(x,y)$ is an acyclic set, where $M : X \times Y \multimap X$ is defined by*

$$M(x,y) = \{u \in S(x) \mid \varphi f(x,y,u) = \operatorname{Min} \varphi f(x,y,S(x))\}.$$

Then there exist an $\bar{x} \in S(\bar{x})$, and a $\bar{y} \in T(\bar{x})$ such that $f(\bar{x},\bar{y},\bar{x}) \in w\operatorname{Min}_D f(\bar{x},\bar{y},S(\bar{x}))$, that is, $f(\bar{x},\bar{y},u) - f(\bar{x},\bar{y},\bar{x}) \notin -\operatorname{int} D$ for all $u \in S(\bar{x})$. Furthermore, if $f(x,y,x) \in D$ for all $(x,y) \in X \times Y$, then there exist an $\bar{x} \in S(\bar{x})$ and a $\bar{y} \in T(\bar{x})$ such that $f(\bar{x},\bar{y},u) \notin -\operatorname{int} D$ for all $u \in S(\bar{x})$.

Proof. By Lemma 7, we see that φf is continuous on $X \times Y \times X$. Then all the conditions of Theorem 7 are satisfied. It follows from Theorem 7 that there exist an $\bar{x} \in S(\bar{x})$ and a $\bar{y} \in T(\bar{x})$ such that

$$\varphi f(\bar{x},\bar{y},\bar{x}) = \operatorname{Min} \varphi f(\bar{x},\bar{y},S(\bar{x})).$$

That is $\varphi f(\bar{x},\bar{y},\bar{x}) \leqslant \varphi f(\bar{x},\bar{y},u)$ for all $u \in S(\bar{x})$. Since $\varphi \in Z^{*} \setminus \{0\}$, it follows that

$$f(\bar{x},\bar{y},u) - f(\bar{x},\bar{y},\bar{x}) \notin -\operatorname{int} D \quad \text{for all } u \in S(\bar{x}).$$

Furthermore, if $f(x,y,x) \in D$ for all $(x,y) \in X \times Y$, then $f(\bar{x},\bar{y},u) \notin -\operatorname{int} D$ for all $u \in S(\bar{x})$. □

Concluding Remarks. In this remark, we will show that relationship between (QEP)′ and vector quasi-variational inequalities for fuzzy mappings or the relationship between (GQEP) and generalized vector quasi-variational inequalities for fuzzy mappings. Let X and Y be topological spaces and Z be a real t.v.s. with closed convex cone D. A fuzzy set on Z is a function with domain Z and values in $[0,1]$, we denote the collection of all fuzzy sets on Z by $\mathscr{F}(Z)$. A mapping $f : X \to \mathscr{F}(Z)$ is called a fuzzy mapping. If $f : X \to \mathscr{F}(Z)$, then $f(x)$ (denoted by f_x) is a fuzzy set on Z, and $f_x(z)$, $z \in Z$ is called the degree of membership of x in f_x. Let A be a fuzzy set in $\mathscr{F}(Z)$ and $\alpha \in (0,1]$, the set $A_\alpha = \{z \in Z : A(z) \geqslant \alpha\}$ is called an α-cut set of A.

Let $s : X \times Y \to \mathscr{F}(X)$, $t : X \to \mathscr{F}(Y)$ and $f : X \times Y \times X \to \mathscr{F}(Z)$ and $g : X \times Y \to \mathscr{F}(Z)$ are fuzzy mappings, α, β and γ are constants in $[0,1]$. We consider the vector quasi-variational inequalities for fuzzy mappings (VQV-IFP) and the generalized vector quasi-variational inequalities for fuzzy mappings (GVQVIFP) as follows:

(VQVIFP): Find $\bar{x} \in (s_{\bar{x}})_\alpha$ such that $z \notin (-\operatorname{int} D)$ for all $z \in (g_{\bar{x},u})_\beta$ and for all $u \in (s_{\bar{x}})_\alpha$.

(GVQVIFP): Find $\bar{x} \in (s_{\bar{x}})_\alpha, \bar{y} \in (t_{\bar{x}})_\gamma$ such that $z \notin (-\operatorname{int} D)$ for all $z \in (f_{\bar{x},\bar{y},u})_\beta$ and all $u \in (s_{\bar{x}})_\alpha$.

As in [4], if we define $S : X \multimap X$ by $S(x) = (s_x)_\alpha$,

$$G : X \times X \multimap Z \text{ by } G(x.y) = (g_{x,y})_\beta,$$

$$F : X \times Y \times X \multimap Z \text{ by } F(x,y,u) = (f_{x,y,u})_\beta. \tag{1}$$

$$T : X \multimap Y \text{ by } T(x) = (t_x)_\gamma,$$

then the vector quasi-variational inequalities for fuzzy mappings (VQVIFP) will reduce to the problem (QEP)′ and the generalized vector quasi-variational inequalities for fuzzy mappings (GVQVIFP) will reduce to (GQEP)′. Hence the study of (QEP)′ and (GQEP)′ have some applications to the study of the problems (VQVIFP) and (GVQVIFP).

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References

[1] J.P. Aubin, A. Cellina, Differential Inclusion, Springer, Berlin, 1994.
[2] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Students 63 (1994) 123–145.
[3] F.E. Browder, The fixed point theory of multi-valued mapping in topological vector space, Math. Ann. 177 (1968) 283–301.
[4] S.S. Chang, G.M. Lee, B.S. Lee, Vector quasi-inequalities for fuzzy mappings I, Fuzzy Sets and Systems 87 (1997) 307–315.
[5] T.H. Chang, C.L. Yen, KKM properties and fixed point theorems, J. Math. Anal. Appl. 203 (1996) 224–235.
[6] O. Hadžić, Fixed point theory in topological vector spaces, Univ. of Novi Sad, Novi Sad, 1984.
[7] C.D. Horvath, Contractibility and generalized convexity, J. Math. Anal. Appl. 156 (1991) 341–351.
[8] V. Klee, Leray-Schauder theory without local convexity, Math. Ann. 141 (1960) 286–297.
[9] J.E. Klein, A. Thompson, Theory of Correspondence, Wiley, 1984.
[10] B.S. Lee, G.M. Lee, S.S. Chang, Generalized vector variational inequalities for multifunctions, Proceedings of Workshop on Fixed Point Theory, June 1997, Poland.
[11] L.J. Lin, Z.T. Yu, Fixed point theorems of KKM type maps, Nonlinear Anal. TMA 38 (1999) 265–275.
[12] L.J. Lin, On generalized loose saddle point theorems for set valued maps, Proceedings of Nonlinear Analysis and Convex Analysis, World Scientific, Niigata, Japan, 1999.
[13] D.T. Luc, Theory of Vector Optimization, Lecture Notes in Economics and Mathematical, Systems, Vol. 319, Springer, Berlin, 1989.
[14] S. Park, Fixed points and quasi-equilibrium problems, Math. Comput. Modelling, to appear.
[15] K.K. Jan, J. Yu, X.Z. Yuan, Existence theorems for saddle points of vector-valued maps, J. Optim. Theory Appl. 89 (1996) 731–747.
[16] T. Tanaka, Generalized quasiconvexities, cone saddle points and minimax theorems for vector valued functions, J. Optim. Theory Appl. 81 (1994) 355–377.
[17] H. Weber, Compact convex sets in non-locally convex linear spaces, Note Math. 12 (1992) 271–289.